VOLUME 81 PAPER No. 798

PROCEEDINGS

AMERICAN SOCIETY
OF
CIVIL ENGINEERS

SEPTEMBER, 1955

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ENGINEERING MECHANICS
DIVISION

{Discussion open until January 1, 1956}

Headquarters of the Society
33 W. 39th St.
New York 18, N. Y.

PRICE $0.50 PER COPY

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This paper was published at 1745 S. State Street, Ann Arbor, Mich., by the American Society of Civil Engineers. Editorial and General Offices are at 33 West Thirty-ninth Street, New York 18, N. Y.

SOIL MECHANICS AND WORK-HARDENING THEORIES OF PLASTICITY

D. C. Drucker,[1] A.M. ASCE, R. E. Gibson,[2] and D. J. Henkel[3]

SUMMARY

Soils having cohesion and internal friction are often treated as perfectly plastic solids. A consistent approach was proposed previously on the basis of the mathematical theory of perfect plasticity and several interesting results were obtained. However, so idealized a treatment will often result in a marked difference between prediction and experimental fact. In particular, the strong dependence of the volume change under shearing action on the prior history of the soil cannot be taken into account properly. The approach suggested now is to treat soil as a work-hardening material which may reach the perfectly plastic state. A rather remarkable qualitative agreement is then obtained with the known behavior of soils in triaxial tests. Additional study along similar lines seems most promising.

INTRODUCTION

There are many problems of soil mechanics in which the soil is considered to behave as a perfectly plastic material obeying a Coulomb criterion, Fig. 1. Stability of slopes, bearing capacity of footings, and pressures on retaining walls are some typical examples. The familiar Rankine stress fields for active and passive pressure, and the much more elaborate work of Meyerhof, Brinch Hansen, and others(1-4)[4] are based essentially upon the assumption of perfect plasticity. The implications of this basic assumption are far reaching and often unexpected as was brought out by Drucker and Prager.(5) In particular it was shown that any generalization of the Coulomb criterion, Fig. 2, which lies properly within the framework of perfect plasticity theory (5,6) must require a volume expansion to accompany shearing action in a soil with a friction angle, ϕ. Whenever the concepts of plasticity theory are applied in calculations of the stress field, the necessary consequences with respect to the displacement pattern can not be ignored. Any lack of agreement between physical reality and the prediction of a self-consistent theory requires re-examination of the underlying hypotheses of the theory, not an inconsistent use of the theory.

Although the general concept of plasticity is accepted in soil mechanics, the relation between stress and strain in the plastic range involves terminology and special concepts which are not in common use. A brief description,

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therefore, will be given of the salient features of the theory of stress-strain relations in plasticity using a familiar two-dimensional state of stress in a metal sheet or thin-walled tube for illustration. Suppose the normal stresses σ_x and σ_y are principal stresses and the third principal stress is zero. The state of stress then may be represented by a point, termed a stress point, on a plot of σ_x vs. σ_y, Fig. 3a. As the state of stress is changed, the resulting strains may be purely elastic or may be elastic-plastic. The curve which encloses elastic stress points, that is the curve which contains the stress points which can be reached without plastic deformation taking place is called the yield curve. In an ordinary material, i.e., a work-hardening material, the stress point must go outside the yield curve for plastic deformation to occur. In ordinary language, the yield point must be exceeded. A new yield curve is established, Fig. 3b, which depending upon the metal and the stress path may or may not resemble the old. Subsequent changes in stress which move the stress point about inside the new curve will again cause elastic changes only.

It is convenient for purposes of description and understanding to superpose strain coordinates on the stress coordinate axes to permit plotting strain and stress on the same diagram. The total strain is made up of an elastic and a plastic component. Strain increments may likewise be decomposed into their elastic and plastic parts. Elastic strain increments can be computed from the stress increments using the elastic stress strain relation. Plastic strain increments alone are shown on Fig. 3b. The theory of plasticity requires that each infinitesimal plastic strain increment $\Delta\epsilon^p$ can be plotted as a vector normal to the existing yield curve at a smooth point and between adjacent normals at a corner.(7) As shown, the x-component of the vector $\Delta\epsilon^p$ is the increment in the plastic strain in the x-direction, $\Delta\epsilon_x^p$, and the y-component is $\Delta\epsilon_y^p$. It is the ratio of the plastic strain increments $\Delta\epsilon_y^p/\Delta\epsilon_x^p$ which is determined or bracketed by the normality condition. At a smooth point, therefore, the ratio of the plastic strain increments is independent of the stress increment ratio. The actual magnitudes of the strain increments depend upon the amount of loading.

The extension to more general states of stress requires an additional coordinate axis for each additional component of stress. Yield curves become yield surfaces in a 3 or higher dimensional space but the picture is essentially the same. The plastic strain increment vector is normal to the existing yield surface at a smooth point or is between adjacent normals at a corner or pointed vertex.

Another important feature of the theory of plasticity is that the fundamental definitions of work-hardening and perfect plasticity lead to the requirement that all yield surfaces must be convex.(7) Convexity means that a line between two points on the surface does not pass outside the region bounded by the surface. If the state of stress is changed from one point inside the yield surface to another in the most direct manner (all the components of stress difference are kept in ratio) no plastic deformation will occur. The straight line AB of Fig. 3b lying wholly within the yield curve illustrates this feature of plasticity theory.

Perfect plasticity is a limiting case of work-hardening in which the subsequent yield surfaces all come closer and closer to the original yield surface. In the limit there is just the one yield surface and the stress point cannot go outside it. Plastic deformations can occur when the state of stress is at yield (the stress point is on the yield surface).

Returning now to soil mechanics, if the Coulomb criterion is assumed to

represent a yield surface for a perfectly plastic solid, simple shear along a plane requires a motion away from the plane as illustrated in Fig. 4a. This is so, because as has been stated the concept of perfect plasticity leads directly to the normality of $\Delta\epsilon^{p}$, Fig. 4b. The vertical component of $\Delta\epsilon^{p}$ represents the increment of plastic shearing strain, $\Delta\gamma^{p}$. The horizontal component, which is always associated with the vertical, points in the tensile stress direction and so represents a plastic extension increment $\Delta\epsilon_{e}^{p} = \Delta\gamma^{p}\tan\phi$.

Although several interesting results of practical importance are obtained by the application of the theory of perfectly plastic bodies to soils, the theory obviously has severe limitations. In particular, experimental evidence indicates that the predicted dilation is usually larger than that found in practice. Also, as is well known, some materials decrease in volume rather than increase when shearing proceeds. Such discrepancy between the prediction of simple theory and physical reality is not surprising. Perfect plasticity is an extreme idealization for metals and may well be more so for soils. Furthermore, there are fundamental differences between plastic and frictional systems[7,8] and all soils are frictional systems to some degree.

Despite the difficulties, it is possible that the behavior of soil can be described fairly well by a theory of plasticity. Metals are really complicated inhomogeneous, anisotropic materials, when looked at in detail, perhaps as complicated as soils. Nevertheless, the behavior of metal is often described reasonably well, in the engineering sense, by stress-strain relations of the work-hardening type for homogeneous material with initial isotropy.

Some Physical Facts and their Significance

There are a number of phenomenon which must be taken into consideration in any reasonably complete theory of plasticity for soil. One of the most important is the plastic component of volume change $(v_o-v)^p$ which takes place when the hydrostatic or equal all-around effective pressure is increased. The effective stress, which controls the deformation of the soil[1] is the difference between the applied stress and the pore water pressure. Consider, for example, a specimen of fully saturated clay subjected to gradually increasing all around pressure p, the test being carried out under fully drained conditions. In drained tests the pore water pressure is zero when equilibrium is established. The effective and applied stresses are then equal. A typical relation between the volume decrease and the pressure p is shown schematically in Fig. 5. Effective stress is plotted in Fig. 5 and all of the figures. If the pressure is reduced at some stage A of loading, the sample swells to R. If then the pressure is increased again to A, the relation thereafter for volume decrease under increasing pressure is closely similar to what would have been obtained had no unloading—reloading cycle been carried out. If the hysteresis loop is averaged, this is clearly the equivalent of a work-hardening stress-strain relation in which the non-linearity of the elastic recovery may be quite pronounced. A similar type of curve is obtained for loose sand[9] but the volume changes are much smaller than for clay. Certainly, if this type of behavior is to be taken into account, the assumption of perfect plasticity, with an extended Coulomb criterion for a yield surface, is not suitable. The yield surface must cut the axis of the cone because yielding does occur for a state of stress represented by a point on the axis of the cone of Fig. 2. Also, the behavior is one involving work-hardening and not perfect plasticity because continuing plastic volume decrease requires increasing pressure.

To be more explicit, consider the triaxial test depicted in Fig. 6 with

$\sigma_2 = \sigma_3$ and in which σ_1 and σ_2 may be varied independently. Note that in all this treatment, the soil mechanics convention is followed of compressive stress positive and tensile stress negative. Decrease in volume is likewise plotted as positive. The opposite convention was employed in references[5] and[6]. With $\sigma_2 = \sigma_3$, the Coulomb criterion may be interpreted as the section of the cone of Fig. 2 by the plane $\sigma_2 = \sigma_3$ as in Fig. 7. Points on the axis of the cone represent hydrostatic pressure $\sigma_1 = \sigma_2 = \sigma_3$. The abscissa is labelled $\sigma_2\sqrt{2}$ because $\sigma_2 = \sigma_3$ and $\sigma_2\sqrt{2}$ is the magnitude of their vector sum or the distance along the line at 45^0 to the σ_2 and σ_3 axes in the plane $\sigma_1 = 0$.

Suppose that the soil has been consolidated to point A on Fig. 5. The yield curve which has been established for the material must pass through the corresponding hydrostatic pressure point A of Fig. 7. The precise shape of the yield curve itself is a matter for much additional research and study. As has already been stated, the yield curve, or in general the yield surface, must be convex[7] so that with but a few points and features established its rough outline is known. It can perhaps be approximated most simply by two straight lines and a circular arc closure as on curve 1 of Fig. 8a. This simple picture corresponds to placing a spherical cap on the open end of the cone of Fig. 2. As the hydrostatic pressure increases to B, the yield surface may change to a similar curve 2 of Fig. 8a. It is likely that the cone which is established as part of the new yield surface will be larger than the previous cone.

As described in the introduction, the concept of work-hardening, and of perfect plasticity as well, leads to the normality of the plastic strain increment vector. The ratio of the components of plastic strain increment is given by the direction of the normal to the yield surface. The components of the normal to the yield curves of Fig. 8 represent $\Delta\epsilon_2^p\sqrt{2}$ and $\Delta\epsilon_1^p$. Again the factor $\sqrt{2}$ appears in the horizontal coordinate because $\Delta\epsilon_2^p = \Delta\epsilon_3^p$ and $\Delta\epsilon_2^p\sqrt{2}$ is their vector sum. Fig. 8 illustrates several loading paths and the plastic strain increments which result. Two quite different pictures are drawn. Fig. 8a is the simpler one which in analogy to metal plasticity ignores the changes in the yield surface which may accompany elastic unloading and reloading. In a soil, the elastic changes have much more significance. Release of pressure results in an elastic volume increase which in the drained test means an increase in water content. Yield strength is affected by water content so that the yield surface will change as the stress point moves about inside it and elastic strains only take place. Fig. 8b shows schematically the effect of the increase in water content upon unloading from B.

Unloading hydrostatically from B to C and then increasing σ_1 to C' and a little beyond causes a plastic volume increase. In a very rough way the picture corresponds to Fig. 4. Another path BDD' where the tangent at D' is parallel to the axis OB of the cone leads to zero plastic volume change, almost as for a frictionless material. Note that the ratio OD/OB is much smaller in Fig. 8b than in Fig. 8a. All paths of the type described which are to the left of D' give a plastic volume increase, all to the right give a plastic volume decrease. Path BEE' is a special case of a plastic increase in volume with $\Delta\epsilon_2^p = 0$. Therefore, there will be a plastic increase in diameter of the triaxial specimen for paths to the left of E' and a plastic decrease for paths to the right.

The paths which have been described are easily produced in the triaxial

test and some typical data is given in Fig. 9. It is important to keep in mind that plastic volume changes only are indicated on Fig. 8. Elastic changes will accompany the plastic and they must be calculated and then subtracted from the total volume change to give the plastic change. The qualitative agreement with the prediction of the simple theory is indeed remarkable. Plastic volume expansion, contraction, and something close to the transition case of no change are all illustrated as plastic deformation begins.

As plastic deformation proceeds, the rate of change of volume does not generally remain constant. This means that as the stress σ_1 is increased in these tests, and the point representing the state of stress moves outside the previously established yield surface, subsequent yield surfaces will alter in significant ways. Much more data must be accumulated and analyzed but it is interesting to speculate that the subsequent yield surfaces tend to rotate the normal to a perpendicular position with respect to the axis of the cone, Fig. 10. The rate of plastic volume increase or decrease then would become zero. In a sense, a perfectly plastic state may be approached in which the yield surface is locally parallel to the axis of the cone and to a first approximation no further hardening takes place. The shear stress required for continued plastic deformation will vary with the pressure but there is now no necessary volume expansion.

A comment is needed on the curves obtained for high overconsolidation. As seen in Figs. 9c,d there is a drop in strength as deformation proceeds. This may be a consequence of frictional action and may not fit quite properly into existing plasticity theory.[8] The normality condition may, therefore, be an approximation for curves 2 and 3 in Fig. 10a.

The Coulomb Criterion

The concept of the Coulomb criterion as a limiting condition for soil still applies but in a different sense. It is not necessarily true, with the point of view now advanced, that the straight lines of Figs. 1, 7, 8 are yield curves for a perfectly plastic material although at times it may be convenient to consider them so along with a reasonable closing circular arc. Instead it is necessary to think in terms of a succession of yield surfaces, Fig. 11. At the failure condition the stress point may be on the end cap and the plastic deformation may involve volume decrease or increase depending upon the prior history of loading. If the failure point is on the end cap, the Coulomb criterion is as shown by the dotted line of Fig. 11 rather than by the outer line. The difference is not very large, except in its implication of volume change, and much additional experimental information must be examined to obtain a clear answer. As the limit line is a little indefinite and as perfect plasticity following work-hardening is an over-simplification, agreement in fine detail cannot be expected.

CONCLUSION

Much more remains to be done before it can be said with any certainty that practical soil mechanics problems can be solved with a work-hardening theory of plasticity. The main advantage, in any case, lies in obtaining an understanding of soil behavior. Such an understanding will be of enormous help in design no matter how difficult it may be to obtain exact solutions. The new approach seems to offer considerable possibilities in this direction as it gives qualitative agreement with known and previously unexplained experimental data from

triaxial tests. Again it should be borne in mind that any simple yield surfaces of the type of Figs. 8, 11 cannot explain all phenomena that will be observed. In particular, the circular end caps are certainly not the proper shape. Other convex yield surfaces should be tried. The theory of stress-strain relations for work-hardening materials is so general that much more can be accomplished with sufficient thought and much additional experimental information.

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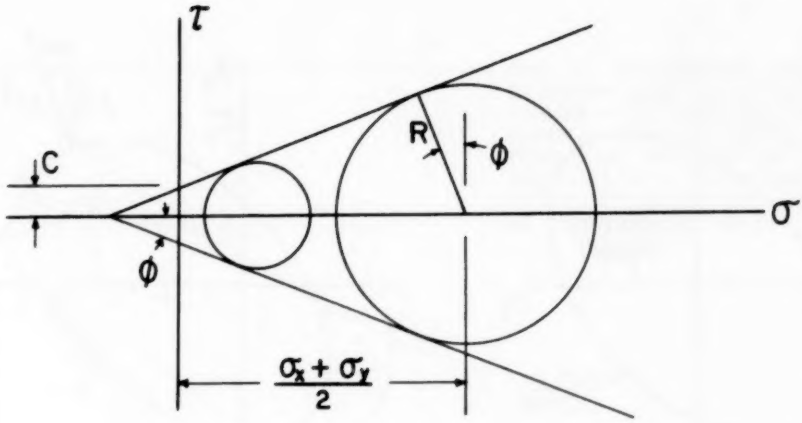


Fig. 1 Coulomb criterion: $R = c \cos\varphi + \frac{\sigma_x + \sigma_y}{2} \sin\varphi$

Note: Positive σ denotes compression

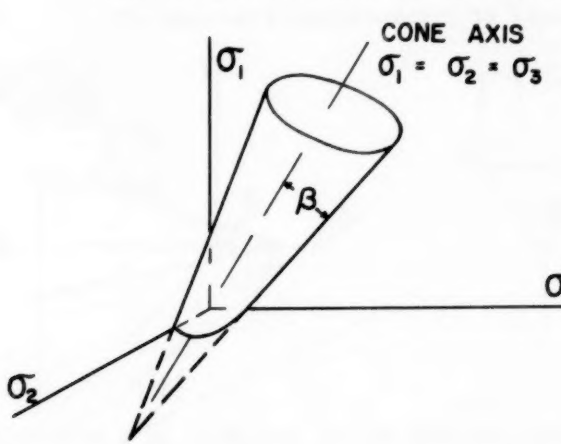


Fig. 2 Mises type extension of Coulomb criterion

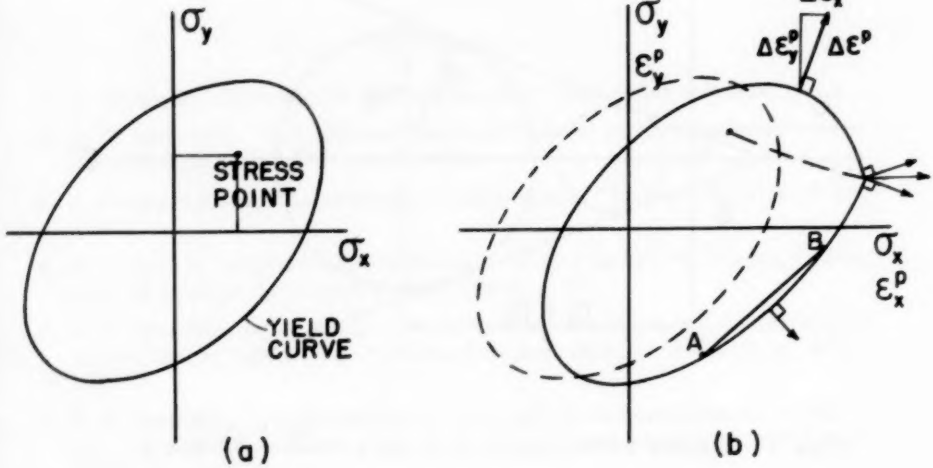


Fig. 3 Concepts of yield surface and normality

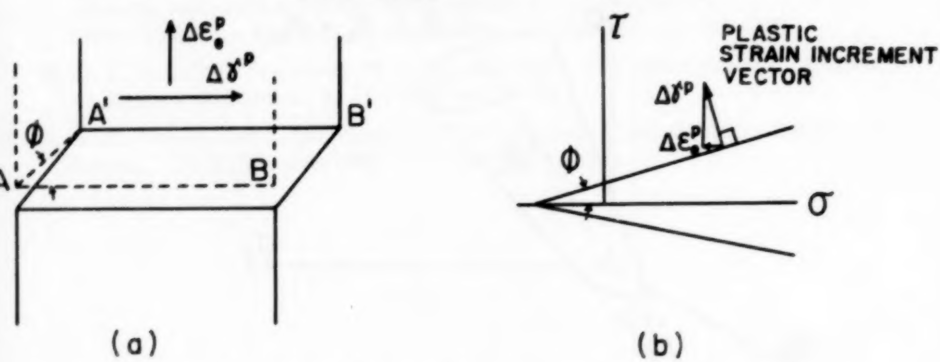


Fig. 4 "Simple" shear produces volume expansion in a perfectly plastic material with a Coulomb yield condition

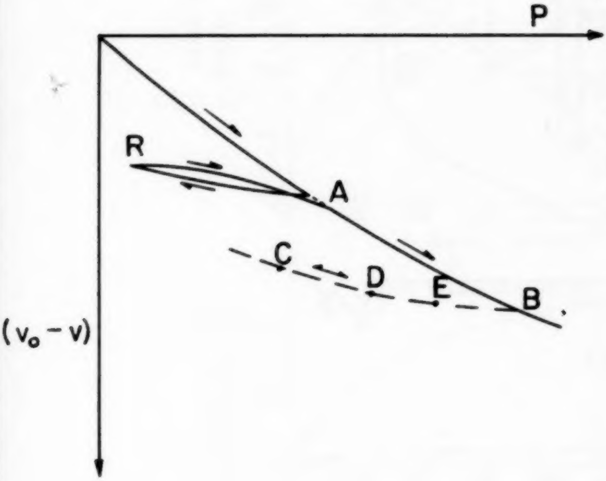
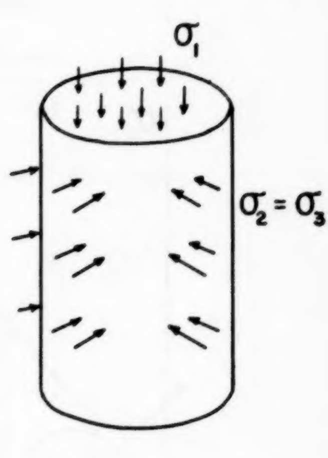


Fig. 5 Hydrostatic pressure produces plastic volume change -- a work-hardening curve. (The unloading and reloading loop may be replaced to a first approximation by an elastic line as shown dashed at B)

Fig. 6 Triaxial test



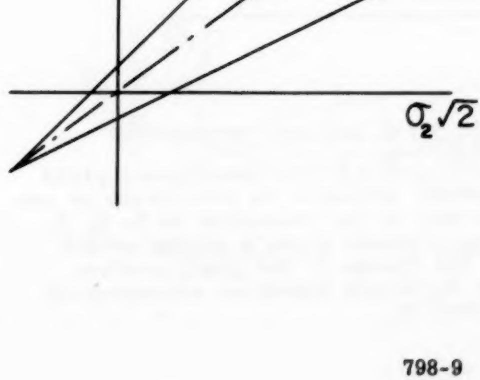


Fig. 7 Section of cone of Fig. 2 by plane $\sigma_2 = \sigma_3$

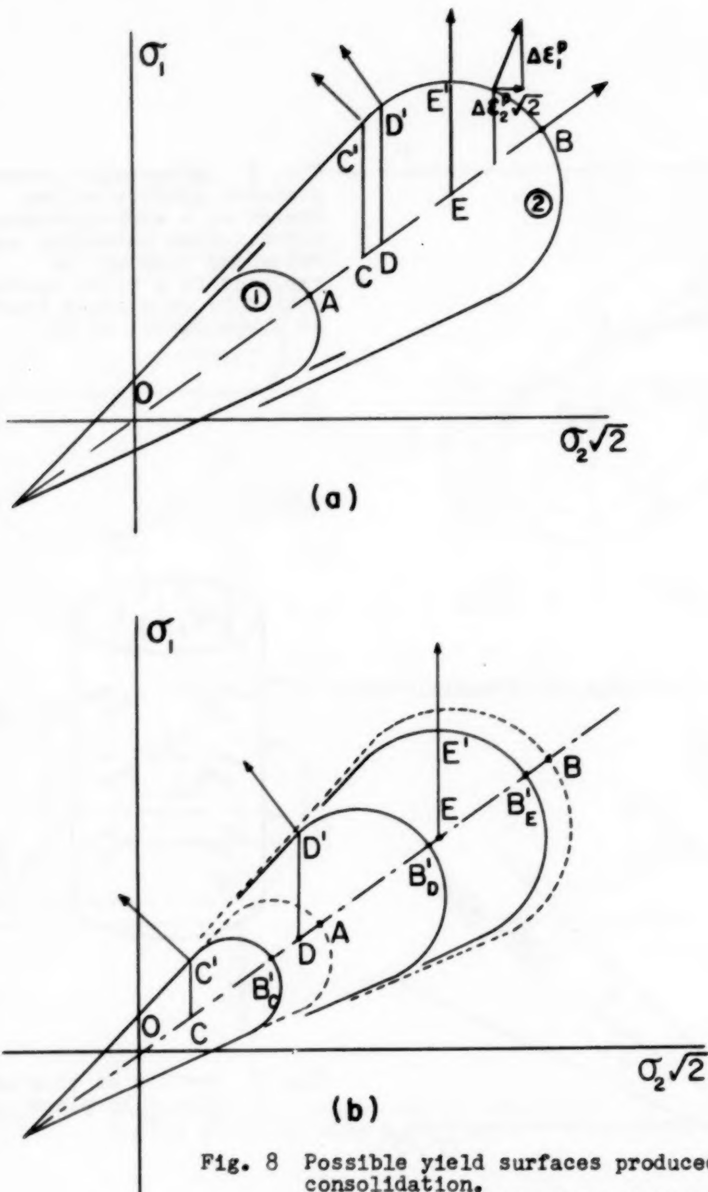
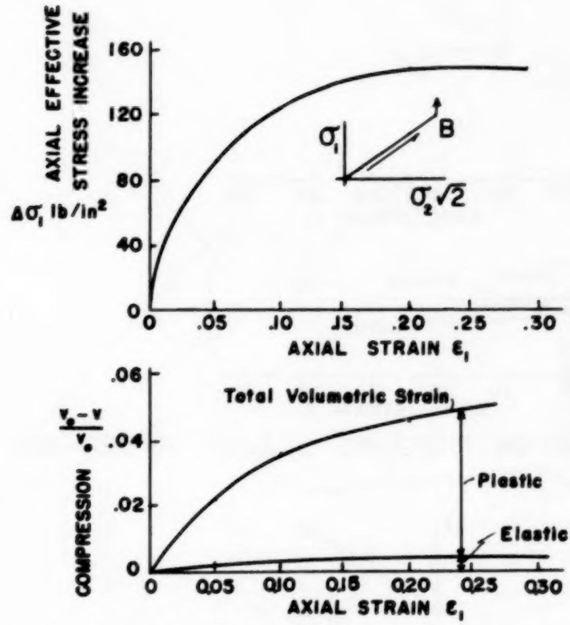
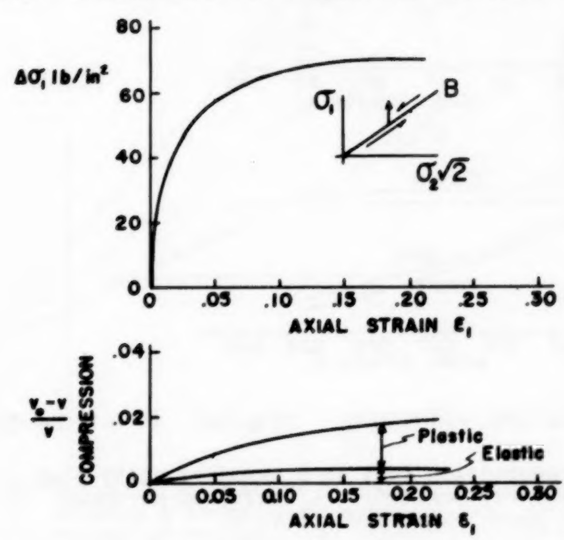


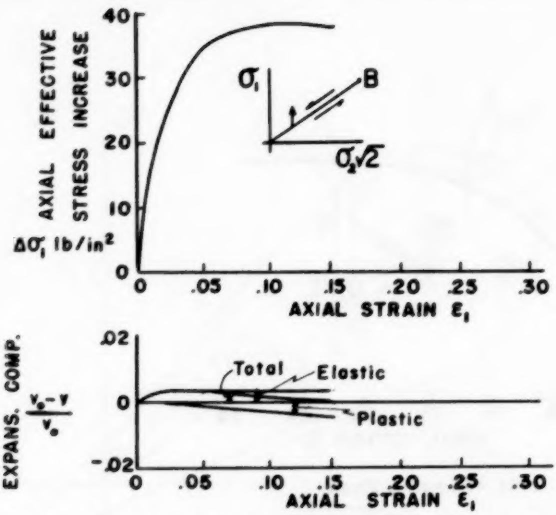
Fig. 8 Possible yield surfaces produced by consolidation.
(a) For simplicity the established yield surface through B is considered as unchanged by the unloading to E, D, C.
(b) A qualitative picture of the effect of the change in the yield surface due to volume expansion accompanying unloading



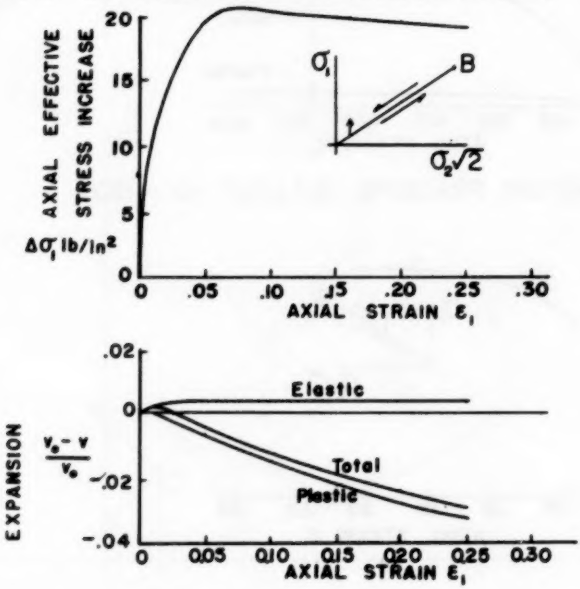
(9a) CONSOLIDATION PRESSURE 120 lb/in² (0 → 120)



(9b) CONSOLIDATION PRESSURE 60 lb/in² (0 → 120 → 60)



(9c) CONSOLIDATION PRESSURE 30 lb/in² (0→120→30)



(9d) CONSOLIDATION PRESSURE 15 lb/in² (0→120→15)

Fig. 9 **Drained compression test on a remolded silty clay from Haslemere, England. σ_1 increased as shown from some point on axis OB, Fig. 8.**

(a) From point B -- Plastic volume decrease quite large compared with axial compression
(b) From point halfway back from B -- Moderate plastic volume decrease
(d) From point 3/4 back from B -- Very small plastic volume increase
(d) From point 7/8 back from B -- Large volume increase

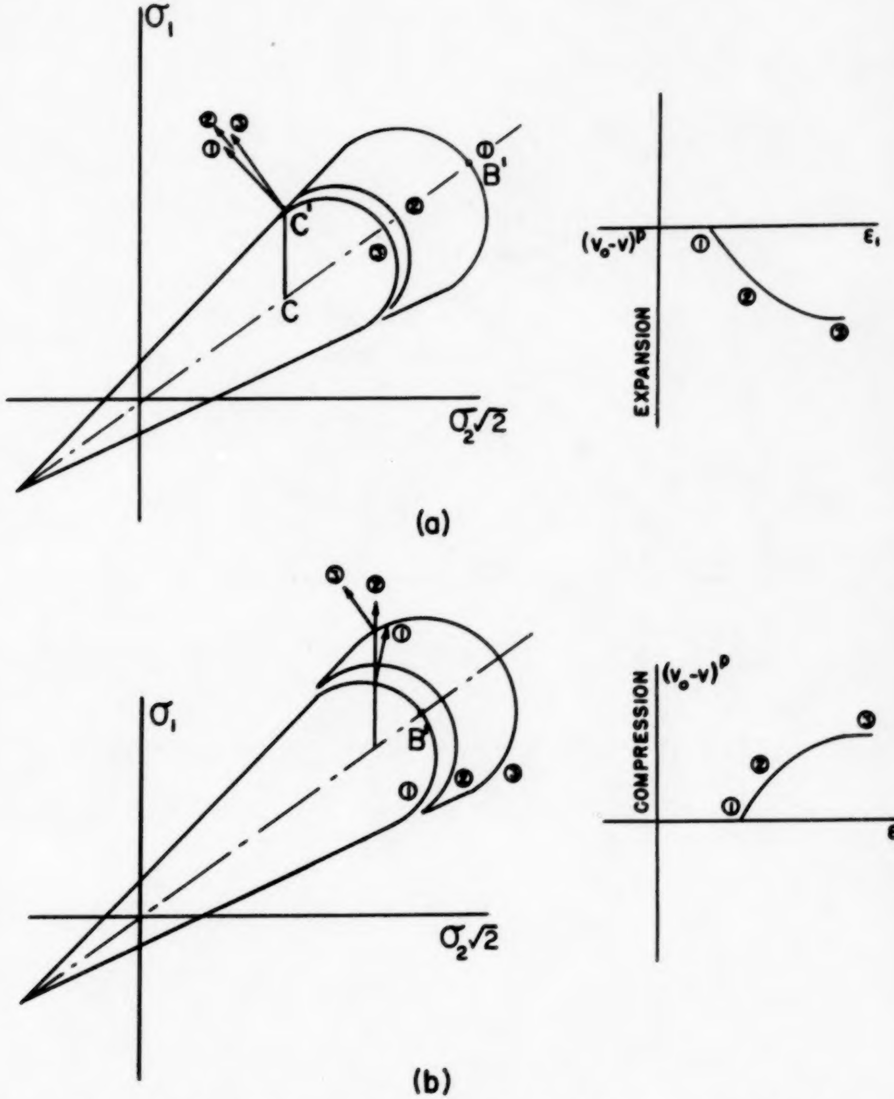


Fig. 10 Possible successive yield surfaces in a triaxial test showing a tendency to approach zero plastic rate of change of volume. (a) Expansion. (b) Compression.

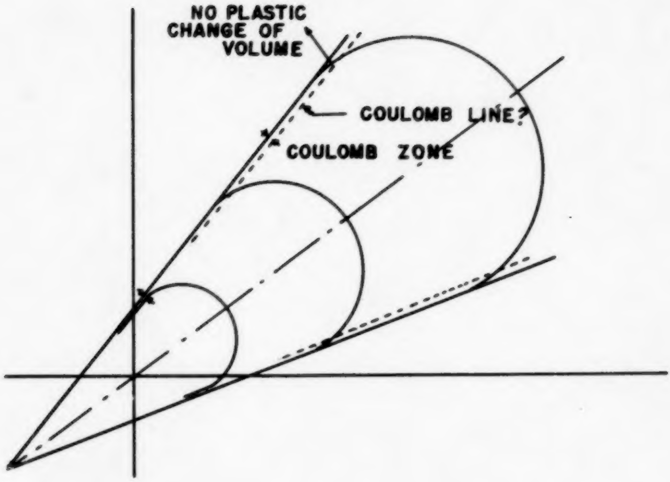


Fig. 11 Possible meaning of the Coulomb criterion